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$c_M < 1$ string theory as a constrained topological sigma model

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Abstract

It has been argued by Ishikawa and Kato that by making use of a specific bosonization, $c_M = 1$ string theory can be regarded as a constrained topological sigma model. We generalize their construction for any (p, q) minimal model coupled to two dimensional (2d) gravity and show that the energy-momentum tensor and the topological charge of a constrained topological sigma model can be mapped to the energy-momentum tensor and the BRST charge of $c_M < 1$ string theory at zero cosmological constant. We systematically study the physical state spectrum of this topological sigma model and recover the spectrum in the absolute cohomology of $c_M < 1$ string theory. This procedure provides us a manifestly topological representation of the continuum Liouville formulation of $c_M < 1$ string theory.

It has been shown recently from various points of view that $c_M = 1$ string theory has manifestly topological field theoretic descriptions. It was first pointed out in Ref. [1] that a special Kazama-Suzuki coset model is equivalent to $c_M = 1$ matter coupled to 2d gravity. Further arguments in favor of the topological nature of $c_M = 1$ string theory were given by identifying it with a topological sigma model [2], a topological G/G model [3] as well as a topological Landau-Ginzburg model [4,5] with a particular superpotential. The latter identification also clarified the origin of the long suspected integrability structure [5,6] in $c_M = 1$ string theory. It should be pointed out here that in these works the equivalence was established by comparing the cohomology structure as well as by computing some correlation functions which agree with the matrix-model results. A more direct approach, clarifying the reason why the observables can be obtained from a topological model, was taken by Ishikawa and Kato in Ref. [7]. They have shown that by making use of a specific bosonization one can identify $c_M = 1$ string theory with a topological sigma model at the level of Lagrangians rather than at the level of amplitudes.

The topological nature of the Liouville approach to $c_M < 1$ string theory is not as clear as in $c_M = 1$ case. It has long been known that certain topological matter coupled to 2d topological gravity reproduce [8,9] the matrix-model results of $c_M < 1$ string theory. The Landau-Ginzburg formulation and the integrability structure

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in this case are also fairly well-understood [10]. After a considerable amount of effort a family of twisted $N = 2$ superconformal structures have been revealed [11,12,13] in the continuum Liouville approach to $c_M < 1$ string theory indicating a close relationship with some topological field theories. Using this information it became clear why $(1, q)$ models coupled to gravity are topological [12]. But still a manifestly topological representation of the Liouville formulation of $c_M < 1$ string theory remained illusive. An attempt in this direction was made in Ref. [14]. By using a bosonization (which reduced to the topological gravity formulation of Distler [15] as a special case), we found that there is a topological gravity structure in any (p, q) model coupled to 2d gravity. It was noted also that the total BRST charge of the topological gravity is different from the string BRST charge and hence it was not clear how to obtain the full spectrum of $c_M < 1$ string theory in the topological gravity representation.

In this paper, we look at a different bosonization similar to the one found in Ref. [7] for $c_M = 1$ string theory. We generalize the construction for any (p, q) minimal model coupled to 2d gravity and show in analogy that $c_M < 1$ string theory can also be regarded as a topological sigma model [17] where one of the coordinates is identified with a ground ring generator [18,19]. In particular, we show that the energy–momentum tensor and the topological charge of the topological sigma model can be mapped to the energy–momentum tensor and the BRST charge of $c_M < 1$ string theory at zero cosmological constant. This approach also clarifies the origin of the twisted $N = 2$ superconformal algebra in $c_M < 1$ string theory. We have also systematically studied the physical state spectrum of this topological sigma model and found that they coincide with the spectrum in the absolute cohomology of $c_M < 1$ string theory.

The (p, q) minimal models (where $\gcd(p, q) = 1$) coupled to 2d gravity can be described in terms of Coulomb gas representation with the total energy–momentum tensor

$$T(z) = T_M(z) + T_L(z) + T^{\text{gh}}(z) \quad (1)$$

where the matter, Liouville and the ghost energy–momentum tensors are given respectively as

$$T_M(z) = -\frac{1}{2} : \partial \phi_M(z) \partial \phi_M(z) : + i Q_M \partial^2 \phi_M(z) \quad (2)$$

$$T_L(z) = -\frac{1}{2} : \partial \phi_L(z) \partial \phi_L(z) : + i Q_L \partial^2 \phi_L(z) \quad (3)$$

$$T^{\text{gh}}(z) = -2 : b(z) \partial c(z) : - : \partial b(z) c(z) : \quad (4)$$

Here we are working in a free theory with zero cosmological constant and so, we concentrate only in the holomorphic sector. Here $\phi_M(z)$, $\phi_L(z)$ denote the matter and Liouville field with the propagators having the form $\langle \phi_M(z) \phi_M(w) \rangle = \langle \phi_L(z) \phi_L(w) \rangle = -\log(z - w)$. $2Q_M$, $2Q_L$ denote the background charges for the matter and Liouville sector which satisfy $Q_M^2 + Q_L^2 = -2$, since the total central charge of the combined matter-Liouville theory is 26. $(b(z), c(z))$ are the usual reparametrization ghost system having conformal weights 2 and -1 respectively with the operator product expansion (OPE) given as $b(z)c(w) \sim \frac{1}{z-w}$. Since the (p, q) minimal models are characterized by the Virasoro central charge $1 - \frac{6(p-q)^2}{pq}$, the background charges for the matter and Liouville sector can be parametrized as

$$Q_M = \left(\frac{1}{2\lambda} - \lambda \right) \quad (5)$$

$$Q_L = i \left(\frac{1}{2\lambda} + \lambda \right) \quad (6)$$

where $\lambda = \sqrt{\frac{q}{2p}}$. We now define the following four conformal fields,

$$x(z) = : \left[b(z)c(z) - \frac{i}{2\lambda} \left(\partial \phi_M(z) + i \partial \phi_L(z) \right) \right] e^{i\lambda(\phi_M(z) - i\phi_L(z))} : \quad (7)$$

$$\bar{p}(z) = : e^{-i\lambda(\phi_M(z) - i\phi_L(z))} : \quad (8)$$

$$B(z) = : b(z) e^{i\lambda(\phi_M(z) - i\phi_L(z))} : \quad (9)$$

$$C(z) = : c(z) e^{-i\lambda(\phi_M(z) - i\phi_L(z))} : \quad (10)$$

with conformal weights 0, 1, 1, 0 respectively with respect to (1). One can easily verify that the OPEs among these fields are

$$\bar{p}(z)x(w) = -x(z)\bar{p}(w) \sim \frac{1}{z-w} \quad B(z)C(w) = C(z)B(w) \sim \frac{1}{z-w} \quad (11)$$

with the rest of the OPEs being regular. We note that the energy-momentum tensor (1) is symmetric under the interchange $\lambda \leftrightarrow \frac{1}{2\lambda}$, $\phi_M(z) \leftrightarrow -\phi_M(z)$ and $\lambda \leftrightarrow \frac{1}{2i\lambda}$, $\phi_M(z) \leftrightarrow \phi_L(z)$ and so, there exists another bosonization which can be obtained by using this symmetry. In terms of these fields (7)–(10) the energy-momentum tensor (1) turns out to be

$$T(z) = - : \partial x(z) \bar{p}(z) : - : B(z) \partial C(z) : \quad (12)$$

By identifying $\bar{p} \equiv \partial \bar{x}$, where \bar{x} denotes the holomorphic part of the complex conjugate of the coordinate x , we easily recognize that the energy-momentum tensor (12) can be obtained from the topological sigma model (in one complex dimension) action [17]. We also point out that for $\lambda = \frac{1}{\sqrt{2}}$, the value for $c_M=1$ string theory, and $\phi_L(z) \rightarrow -\phi_L(z)$ our bosonization (7)–(10) matches precisely with Eq. (2.3) in Ref. [7]. We would like to mention here that one of the coordinates of the topological sigma model Eq. (7) is nothing but one of the ground ring generators of $c_M < 1$ string theory. This is a true conformal field of weight zero in the sense that it does not contain the $\log z$ term in its expansion in z like a proper sigma model coordinate. In this sense, the topological sigma model in question here is a constrained (zero momentum sector) one.

The nilpotent supersymmetry current of this topological sigma model has the form

$$Q(z) = -C(z)\partial x(z) \quad (13)$$

Substituting (10) and (7) we find

$$Q(z) = : c(z) [T_M(z) + T_L(z) + \frac{1}{2}T^{\text{gh}}(z)] : + \frac{1}{2}\partial \left[\partial c(z) + 2\lambda c(z) \left(i\partial\phi_M(z) + \partial\phi_L(z) \right) \right] \quad (14)$$

Therefore, we conclude from (14), that the topological charge of the topological sigma model is the same as the BRST charge of $c_M < 1$ string theory. The origin of the twisted $N=2$ superconformal algebra in $c_M < 1$ string theory [12,13] can be understood in terms of the corresponding algebra in the associated topological sigma model. It is straightforward to check that the following generators of the topological sigma model

$$T(z) = : \partial x(z) \bar{p}(z) : - : B(z) \partial C(z) : \quad (15)$$

$$G^+(z) = C(z)\partial x(z) - (a_3 - \frac{1}{2})\partial(C(z)x(z)) \quad (16)$$

$$G^-(z) = B(z)\bar{p}(z) \quad (17)$$

$$J(z) = : C(z)B(z) : + (a_3 - \frac{1}{2})[: C(z)B(z) : + : x(z)\bar{p}(z) :] \quad (18)$$

satisfy a twisted superconformal algebra with the associated $N=2$ central charge $c^{N=2} = 6a_3$, where a_3 is a constant parameter. We note that $G^+(z)$ is the topological current (13) modified by a total derivative term, so that, the topological charge is not affected. Also, the $U(1)$ current $J(z)$ has been modified suitably so that they form a closed algebra. For $a_3 = \frac{1}{2}$ these generators were described in the context of topological sigma models

in [20]. Substitution of the topological sigma model fields in terms of the fields in string theory (7)–(10), gives the corresponding generators in $c_M < 1$ string theory,

$$T(z) = -\frac{1}{2} : \partial \phi_M(z) \partial \phi_M(z) : + i Q_M \partial^2 \phi_M(z) - \frac{1}{2} : \partial \phi_L(z) \partial \phi_L(z) : + i Q_L \partial^2 \phi_L(z) \\ - 2 : b(z) \partial c(z) : - : \partial b(z) c(z) : \quad (19)$$

$$G^+(z) = : c(z) [T_M(z) + T_L(z) + \frac{1}{2} T^{\text{gh}}(z)] : + a_1 \partial (c(z) \partial \phi_L(z)) + a_2 \partial (c(z) \partial \phi_M(z)) \\ + a_3 \partial^2 c(z) \quad (20)$$

$$G^-(z) = b(z) \quad (21)$$

$$J(z) = : c(z) b(z) : - a_1 \partial \phi_L(z) - a_2 \partial \phi_M(z) \quad (22)$$

where $a_1 = \frac{1}{4} [i Q_L (2a_3 - 3) - Q_M (2a_3 + 1)]$ and $a_2 = \frac{1}{4} [i Q_M (2a_3 - 3) + Q_L (2a_3 + 1)]$. This formalism, therefore, clarifies the origin of the twisted $N = 2$ superconformal algebra in $c_M < 1$ string theory found in Ref. [13].

In order to find the physical state spectrum, we note that the topological charge of the topological sigma model is given by¹

$$Q = - \oint dz C(z) \partial x(z) = \sum_m m C_{-m} x_m \quad (23)$$

Physical states of the topological sigma model are the states which are in the kernel of this charge modulo its image. First we note that the vacuum of the topological sigma model is characterized by the following regularity conditions,

$$B_n |0\rangle_t = \bar{p}_n |0\rangle_t = 0 \quad \text{for } n \geq 0 \\ C_n |0\rangle_t = x_n |0\rangle_t = 0 \quad \text{for } n \geq 1 \quad (24)$$

which follow from their respective conformal weights. Using (24) it is easy to check that $|0\rangle_t$ is physical with respect to (23). Since the topological charge (23) is the same as the string BRST charge we identify $|0\rangle_t$ as the $SL(2, \mathbb{C})$ invariant vacuum and so we drop the subscript ‘t’ subsequently. The physical spectrum of the topological sigma model can be built by applying the physical modes on this vacuum. In order to determine the physical modes we note that

$$[Q, x_n] = 0 \quad (25)$$

$$[Q, \bar{p}_n] = -n C_n \quad (26)$$

$$\{Q, B_n\} = n x_n \quad (27)$$

$$\{Q, C_n\} = 0 \quad (28)$$

where $[]$ and $\{ \}$ indicate commutator and anticommutator respectively. It is clear from (25)–(28) that for $n \neq 0$, all the modes are non-physical as they form “quartets”, but for $n = 0$ this is no longer true and x_0 , \bar{p}_0 , B_0 and C_0 are the physical modes. Since B_0 and p_0 annihilate the topological vacuum, the physical states would have the form $C_0^\epsilon x_0^n |0\rangle$, where $\epsilon = 0$ or 1 and $n = 0, 1, 2, \dots$. After a few simple calculations we identify these states with some of the states of $c_M < 1$ string theory [19,21–24] as follows,

$$x_0^n |0\rangle = : x^n(0) : |0\rangle \quad (29)$$

¹ A conformal field of weight h is expanded in terms of modes as $\phi(z) = \sum_n \phi_n z^{-n-h}$.

$$C_0|0\rangle = :c(0)e^{-i\lambda(\phi_M(0)-i\phi_L(0))}:|0\rangle \quad (30)$$

$$C_0x_0|0\rangle = -\left(\partial c(0) - \frac{1}{2\lambda}c(0)\partial\phi_L(0) + \frac{i}{2\lambda}c(0)\partial\phi_M(0)\right)|0\rangle = a(0)|0\rangle \quad (31)$$

and in general

$$C_0x_0^n|0\rangle = :a(0)x^{n-1}(0):|0\rangle \quad (32)$$

We notice that the physical state (30) is a tachyonic state whose matter momentum lies on the edge of the Kac-table. Also, the state

$$a(z) = \left[Q, \frac{1}{2\lambda}(\phi_L(z) - i\phi_M(z))\right] = -\left(\partial c(z) - \frac{1}{2\lambda}c(z)\partial\phi_L(z) + \frac{i}{2\lambda}c(z)\partial\phi_M(z)\right)$$

is a physical state in the absolute cohomology [18] since $\oint dz z b(z) a(0) \neq 0$. So, in this way we recover part of the spectrum of $c_M < 1$ string theory in the absolute cohomology.

In order to obtain the rest of the physical states we recall [25] that in a bosonization, there are more distinct inequivalent representations of vacua in the original theory, known as the picture changed vacua, which should be included in order to compare the physical states of both theories. Since $c_M < 1$ string theory is a bosonized form of the topological sigma model, we recover the rest of the states from the picture changed vacua. The most general picture changed vacuum can be obtained from the bosonization formula (7)–(10) and has the form

$$\begin{aligned} |q_1, q_2\rangle &= :e^{i(q_1Q_M - \lambda q_2)\phi_M(0) + i(q_1Q_L + i\lambda q_2)\phi_L(0) - i(q_1 + q_2)\psi(0)}:|0\rangle \\ &=:e^{i\left(\frac{q_1}{2\lambda} - (q_1 + q_2)\lambda\right)\phi_M(0) - \left(\frac{q_1}{2\lambda} + (q_1 + q_2)\lambda\right)\phi_L(0) - i(q_1 + q_2)\psi(0)}:|0\rangle \end{aligned} \quad (33)$$

Here q_1 and q_2 are some fixed numbers. $\psi(z)$ is a bosonic field obtained from the bosonization of the reparametrization ghosts $b(z) = :e^{i\psi(z)}:$ and $c(z) = e^{-i\psi(z)}$. In terms of modes the picture changed vacuum satisfies

$$x_n|q_1, q_2\rangle = 0 \quad \text{for } n \geq 1 - q_1 \quad (34)$$

$$\bar{p}_n|q_1, q_2\rangle = 0 \quad \text{for } n \geq q_1 \quad (35)$$

$$B_n|q_1, q_2\rangle = 0 \quad \text{for } n \geq q_2 \quad (36)$$

$$C_n|q_1, q_2\rangle = 0 \quad \text{for } n \geq 1 - q_2 \quad (37)$$

Using (34)–(37) we find that the topological charge (23) acting on the picture changed vacuum gives

$$Q|q_1, q_2\rangle = \sum_m m C_{-m} x_m |q_1, q_2\rangle = - \sum_{m=q_1}^{m=-q_2} m x_{-m} C_m |q_1, q_2\rangle \quad (38)$$

So, the vacuum is Q -invariant if $q_1 + q_2 \geq 1$. When $q_1 + q_2 < 1$ an invariant vacuum could be constructed as $C_{-q_2} C_{-q_2-1} \dots C_{q_1+2} C_{q_1+1} C_{q_1} |q_1, q_2\rangle$ since C is a fermionic field. By using $C_n = \oint dz z^{n-1} C(z)$ and Eq. (10), we can easily show that

$$C_{-q_2} C_{-q_2-1} \dots C_{q_1+2} C_{q_1+1} C_{q_1} |q_1, q_2\rangle = |q_1, 1 - q_1\rangle \quad (39)$$

So, the most general Q -invariant picture changed vacuum has the form

$$|q_1, q_2\rangle \quad \text{where } q_1 + q_2 \geq 1 \quad (40)$$

and the vacuum (39) is already contained in (40) as a special case. We also note that in order for this vacuum to be non-trivial (non Q -exact), its conformal weight would have to be zero. Since the conformal weight of the picture changed vacuum $|q_1, q_2\rangle$ is $\frac{1}{2}(q_2 - q_1)(q_1 + q_2 - 1)$ this will happen only when $q_1 = q_2$ or $q_1 + q_2 = 1$. For the first case, both q_1 and q_2 have to be positive integers since $q_1 + q_2 \geq 1$. But, it is not difficult to see that the states of the form $|m, m\rangle$ with m , a positive integer, are all Q -exact. In general, the states of the form $B_0^\epsilon \bar{p}_0^n |m, m\rangle$ where $\epsilon = 0$ or 1 as before and $n = 0, 1, 2, \dots$ can be shown to be Q -exact, since Q anticommutes with B_0 and commutes with \bar{p}_0 . Other physical modes x_0 and C_0 annihilate the vacuum.

It, therefore, follows that the non-trivial physical states can be obtained by applying the physical modes on the picture changed vacuum of the form $|q_1, 1 - q_1\rangle$ for $q_1 > 0$ or $q_1 \leq 0$. When $q_1 > 0$, we note that the physical modes which do not annihilate the vacuum are C_0 and \bar{p}_0 , but all the states of the form $C_0^\epsilon \bar{p}_0^n |q_1, 1 - q_1\rangle$, for $q_1 > 0$ can be shown to be cohomologically trivial except when $q_1 = 1$ and $\epsilon = 0$. In that case, the states lie on the edge of the Kac-table. For other cases, they give combinations of null vectors and Q -exact terms. For $q_1 \leq 0$, the physical modes which do not annihilate the vacuum are B_0 and x_0 . So, in this case, states would be of the form

$$B_0^\epsilon x_0^n | -m, 1 + m\rangle \quad \text{where } m \geq 0 \quad (41)$$

It is easy to see that for $m = 0$, $|0, 1\rangle = C_0|0\rangle$ and so, they do not generate any new state as they have already been obtained in (29)–(32). For $m = 1$, the vacuum itself lies on the edge of the Kac-table and

$$\begin{aligned} x_0 | -1, 2\rangle &= : \left[-\frac{i}{2\lambda} \partial^2 \phi_M(0) + \frac{1}{2\lambda} \partial^2 \phi_L(0) + i\partial^2 \psi(0) + \frac{1}{2} \partial \phi_M(0) \partial \phi_M(0) - \lambda \partial \phi_M(0) \partial \psi(0) \right. \\ &\quad \left. + \frac{1}{2} \partial \phi_L(0) \partial \phi_L(0) + i\lambda \partial \phi_L(0) \partial \psi(0) \right] e^{-\frac{1}{2\lambda} \phi_M(0) + \frac{1}{2\lambda} \phi_L(0) - i\psi(0)} : |0\rangle \\ &= - : a(0) y(0) : |0\rangle \end{aligned} \quad (42)$$

Also,

$$B_0 | -1, 2\rangle = : [i\lambda \partial \phi_M(0) + \lambda \partial \phi_L(0) + i\partial \psi(0)] e^{-\frac{1}{2\lambda} \phi_M(0) + \frac{1}{2\lambda} \phi_L(0)} : |0\rangle = y(0) |0\rangle \quad (43)$$

Here $y(z) = : [b(z)c(z) + \lambda(i\partial \phi_M(z) + \partial \phi_L(z))] e^{-\frac{1}{2\lambda} \phi_M(z) + \frac{1}{2\lambda} \phi_L(z)} :$ is the other ground ring generator of $c_M < 1$ string theory. In general, we find

$$x_0^n | -1, 2\rangle = - : a(0) y(0) x^{n-1}(0) : |0\rangle \quad (44)$$

$$B_0 x_0^n | -1, 2\rangle = : y(0) x^n(0) : |0\rangle \quad (45)$$

For $m = 2$, the vacuum itself again lies on the edge of the Kac-table but

$$x_0^n | -2, 3\rangle = -\frac{1}{2!} : a(0) y^2(0) x^{n-1}(0) : |0\rangle \quad (46)$$

$$B_0 x_0^n | -2, 3\rangle = \frac{1}{2!} y^2(0) x^{n-1}(0) : |0\rangle \quad (47)$$

For a general picture changed vacuum of this type ($m > 0$), we have

$$x_0^n | -m, m+1\rangle = -\frac{1}{m!} : a(0) y^m(0) x^{n-1}(0) : |0\rangle \quad (48)$$

$$B_0 x_0^n | -m, m+1\rangle = \frac{1}{m!} : y^m(0) x^{n-1}(0) : |0\rangle \quad (49)$$

Until now we have seen how to recover all the powers of x and y as well as the operators multiplied by a in the topological model. Now we show how the tachyons of $c_M < 1$ string theory can be obtained as a picture changed vacuum. Let us recall from the results of $c_M < 1$ string theory [24] that the tachyons whose matter momenta lie inside the Kac-table, and whose Liouville momenta satisfy $p_L < Q_L$, can be written in general as

$$: w^{-1}(z) x^{p-m'-1}(z) y^{m-1}(z) := : c(z) e^{i\alpha_{m,m'} \phi_M(z) + i\beta_{m,-m'} \phi_L(z)} : \quad (50)$$

Here $w^{-1}(z) = : c(z) e^{i\alpha_{1,p-1} \phi_M(z) + i\beta_{1,-p+1} \phi_L(z)} :$, m and m' are integers with the restrictions $1 \leq m' \leq p-1$; $1 \leq m \leq q-1$ for (p, q) minimal models coupled to 2d gravity. Also,

$$\alpha_{m,m'} = \frac{1}{2\lambda}(1-m) - \lambda(1-m') \quad (51)$$

$$\beta_{m,m'} = \frac{i}{2\lambda}(1-m) + i\lambda(1-m') \quad (52)$$

It is easy to check that the picture changed vacuum with $q_1 = (1-m) + 2m'\lambda^2$ and $q_2 = m - 2m'\lambda^2$ will have the correct matter and Liouville momenta of the tachyonic state (50). We note that the picture charge for the vacuum associated with the tachyons are fractional and so, none of the modes of x , \bar{p} , B and C are well-defined on the picture changed vacua. Since for these cases $q_1 + q_2 = 1$, the picture changed vacua themselves are Q -invariant but, no new states can be obtained by applying the modes of x , \bar{p} , B and C on the vacua.

This, therefore, completes our analysis how to recover the physical states of ghost number¹ zero (tachyons) and -1 (ground ring generators) of $c_M < 1$ string theory in the associated topological sigma model. Finally we make a comment about the ghost number -2 state $w(0)|0\rangle$, present in $c_M < 1$ string theory [23,24] and whose powers generate the higher ghost number states. The general form of this state is,

$$w(0)|0\rangle = : \mathcal{P}(\partial\phi_M, \partial\phi_L, b, c) e^{i\alpha_{q-1,1} \phi_M(0) + i\beta_{1,p+1} \phi_L(0)} : |0\rangle \quad (53)$$

$$\text{or} = : \mathcal{P}(\partial\phi_M, \partial\phi_L, b, c) e^{i\alpha_{1,p-1} \phi_M(0) + i\beta_{q+1,1} \phi_L(0)} : |0\rangle \quad (54)$$

where \mathcal{P} is a differential polynomial of conformal weight $(p+q-1)$. The form of $w(0)|0\rangle$ would be quite complicated in general, but for small (p, q) values it can be calculated with reasonable effort. In fact, for $(2,3)$ model coupled to gravity, it has the form:

$$w(0)|0\rangle = : \left[2i\partial^2\psi(0) + \partial\psi(0)\partial\psi(0) - \frac{3}{2}i\partial\psi(0)\partial\phi_L(0) + \sqrt{3}\partial^2\phi_L(0) \right] e^{\sqrt{3}\phi_L(0) + i\psi(0)} : |0\rangle \quad (55)$$

where $: e^{i\psi(z)} : = b(z)$. We find that for $(2,3)$ model, this state can be obtained as,

$$w(0)|0\rangle = \frac{1}{16} [23B(0)\partial C(0) - 3\partial B(0)C(0) + 6x(0)\partial\bar{p}(0) + 30\partial x(0)\bar{p}(0) - 9(x(0)\bar{p}(0))^2 + 6B(0)C(0)x(0)\bar{p}(0)] : |-\frac{3}{2}, \frac{1}{2}\rangle \quad (56)$$

where $|-\frac{3}{2}, \frac{1}{2}\rangle$ denotes the picture changed vacuum with picture charge $q_1 = -\frac{3}{2}$ and $q_2 = \frac{1}{2}$. We remark that since $q_1 + q_2 < 1$ for this vacuum, it is not Q -invariant by itself, but the whole combination on the right hand side of (56) is Q -invariant. Also, it is clear that, since the picture charge for this vacuum is fractional, the individual modes of the fields $B(z)$, $C(z)$, $x(z)$ and $\bar{p}(z)$ are not well-defined on this vacuum but the composite fields appeared in (56) are perfectly well-defined. In this sense, $w(0)|0\rangle$ does have a good description in the topological sigma model we have obtained.

It should be pointed out here that, there are, in fact, two sets of w and w^{-1} in $c_M < 1$ string theory and they are necessary in order to have a well-defined product among themselves. These two sets can easily be

¹ Recall that the ghost number of a physical state equals to the ghost number of the corresponding operator minus one [18].

seen to be related by the symmetry $\lambda \leftrightarrow \frac{1}{2\lambda}$, $\phi_M \leftrightarrow -\phi_M$; $\lambda \leftrightarrow \frac{1}{2i\lambda}$, $\phi_M \leftrightarrow \phi_L$ mentioned before. Since the bosonization (7)–(10) does not respect this symmetry, we did not get both sets of the operators in one particular bosonization. But, we notice that the matter sector of the second set of w and w^{-1} belongs to the dual of the first set, so, one would expect them to appear in the dual representation of the picture changed vacuum. We have checked that this is indeed the case.

To conclude, we have shown that any (p, q) minimal model coupled to gravity can be regarded as the bosonized form of a constrained topological sigma model in analogy with the corresponding result for $c_M = 1$ string theory. We have shown that not only the energy–momentum tensor and BRST charge of these two models are identical under this bosonization but also the physical states in both these theories are in agreement. Our approach clarified, as a byproduct, the origin of the twisted $N = 2$ superconformal algebra in $c_M < 1$ string theory. There are, however, some subtleties which we have pointed out. The significance of the ghost number -2 state is not quite clear as it is obtained on a picture changed vacuum which is not Q -invariant. Finally, we mention that, as in the topological gravity formulation of $c_M < 1$ string theory [14], we do not get the restrictions on the ground ring generators of the form $x^{p-1} = y^{q-1} = 0$. These were imposed in $c_M < 1$ string theory such that the matter momenta of the physical states would lie inside the Kac-table. There are, however, differences of opinion about this issue. A different physical state structure has been proposed in Ref. [26] for $c_M < 1$ string theory using the descent equations of the double cohomology i.e. the usual string BRST cohomology and Felder's BRST cohomology. In that case, one does allow all the powers of the ground ring generators modded out by the equivalence relation $x^p \simeq y^q$, as the physical states. How to implement similar procedure in the topological sigma model representation of $c_M < 1$ string theory we obtained is not clear to us.

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